

EXTENDING THE TORELLI MAP TO TOROIDAL COMPACTIFICATIONS OF SIEGEL SPACE

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ABSTRACT. It has been known since the 1970s that the Torelli map $M_g \rightarrow A_g$, associating to a smooth curve its jacobian, extends to a regular map from the Deligne-Mumford compactification \overline{M}_g to the 2nd Voronoi compactification $\overline{A}_g^{\text{vor}}$. We prove that the extended Torelli map to the perfect cone (1st Voronoi) compactification $\overline{A}_g^{\text{perf}}$ is also regular, and moreover $\overline{A}_g^{\text{vor}}$ and $\overline{A}_g^{\text{perf}}$ share a common Zariski open neighborhood of the image of \overline{M}_g . We also show that the map to the Igusa monoidal transform (central cone compactification) is *not* regular for $g \geq 9$; this disproves a 1973 conjecture of Namikawa.

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Introduction

The Torelli map $M_g \rightarrow A_g$ associates to a smooth curve C its jacobian JC , a principally polarized abelian variety. Does it extend to a regular map $\overline{M}_g \rightarrow \overline{A}_g$ between the compactified moduli spaces?

For the moduli space of curves M_g , a somewhat canonical choice of a compactification is provided by the Deligne-Mumford compactification \overline{M}_g , which we fix for the remainder of the paper.

We note in passing that recently other compactifications $\overline{M}_g(\alpha)$ were considered by many authors. These are log canonical models of \overline{M}_g with respect to $K_{\overline{M}_g} + \alpha\delta$, where δ is the boundary. They also have modular interpretation. For example, for $9/11 \geq \alpha > 7/10$, assuming $g \geq 3$, $\overline{M}_g(\alpha)$ is the moduli spaces of curves with nodes and cusps and without elliptic tails. However, the extended map $\overline{M}_g(\alpha) \rightarrow \overline{A}_g$ has no chance of being regular (unless $\overline{M}_g(\alpha) = \overline{M}_g$) because curves of compact type with elliptic tails map to the interior A_g , which has to be suitably modified as well.

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For the moduli space of principally polarized abelian varieties A_g , by [AMRT75] there are infinitely many choices of toroidal compactifications \overline{A}_g^τ , each determined by a fan τ supported on the space of positive semidefinite quadratic forms in g variables, periodic w.r.t. $\mathrm{GL}(g, \mathbb{Z})$, with only finitely many orbits. There are three standard explicit choices for τ , and they all have interesting geometric meanings:

- (1) 1st Voronoi fan = perfect cones τ^{perf} ,
- (2) 2nd Voronoi fan = Delaunay-Voronoi fan = L-type domains τ^{vor} ,
- (3) central cones τ^{cent} .

The first two of these were defined by G. Voronoi in a series of papers [Vor09] on reduction theory of quadratic forms, published posthumously in 1908-9.

The 2nd Voronoi compactification appears in [Ale02] as the normalization of the main irreducible component of the moduli space $\overline{\mathrm{AP}}_g$ of stable semiabelic pairs (X, Θ) which provides a moduli compactification of A_g . On the other hand, by [SB06] the perfect cone compactification $\overline{A}_g^{\mathrm{perf}}$ is the canonical model of any smooth compactification of A_g , if $g \geq 12$ (and also for all g , if considered as stacks and relatively over Satake-Baily-Borel compactification \overline{A}_g^*).

The central cones fan was introduced by Igusa [Igu67]; the corresponding toroidal compactification $\overline{A}_g^{\mathrm{cent}}$ is the normalization of the blowup of the Satake-Baily-Borel compactification \overline{A}_g^* along the boundary (the ‘‘Igusa blowup’’).

The basic question we consider is this: for which choices of a fan τ does the Torelli map $M_g \rightarrow A_g$ extend to a regular map $\overline{M}_g \rightarrow \overline{A}_g^\tau$? For the 2nd Voronoi fan, a positive answer was given by Mumford and Namikawa [Nam76, §18]. This prompted an extensive study of the 2nd Voronoi compactification by Namikawa [Nam76], continued in the construction of the moduli of stable semiabelic pairs $\overline{\mathrm{AP}}_g$ in [Ale02]. The work [Ale04] gives a modular interpretation for the extended Torelli map $\overline{M}_g \rightarrow \overline{\mathrm{AP}}_g$.

Historically, the extension question for the Igusa blowup was the first one to be considered, in a pioneering 1973 paper [Nam73] of Namikawa. There, it is shown that $\overline{M}_g \rightarrow \overline{A}_g^{\mathrm{cent}}$ is regular for low g (the bound $g \leq 6$ is stated without proof), regular on the locus of curves with a planar dual graph, and conjectured that the map is regular for all g .

The question for the perfect cone compactifications was not previously considered, to our knowledge.

In this paper, we prove that the extended map is regular for the perfect cone compactification for all g . Much more than that, we prove that the perfect and the 2nd Voronoi compactifications share a common open neighborhood of the image of \overline{M}_g . Note that in general there is a birational map $\overline{A}_g^{\mathrm{vor}} \dashrightarrow \overline{A}_g^{\mathrm{perf}}$ which does not create new divisors. It is an isomorphism iff $g \leq 3$, and regular for $g \leq 5$. According to [ER01, ER02], this map is not regular for $g \geq 6$. Thus, for higher g the two compactifications are truly different, but we prove that they are equal near the closure of the Schottky locus.

For the central cone compactification, we prove that the extended map is regular for $g \leq 6$ and is *not* regular for $g \geq 9$. Continuing the methods of the present paper, [ALT⁺10] also settled the cases $g = 7, 8$ positively, by a lengthy computation.

The structure of the paper is as follows. In Section 1 we recall the combinatorial data for a toroidal compactification of A_g , and define the fans τ^{perf} , τ^{cent} , τ^{vor} .

In Section 2 we fix the notations for graphs and define *edge-minimizing metrics*, which we abbreviate to *emm*, on the first cohomology group $H^1(G, \mathbb{Z})$ of a graph.

In Section 3 we give a general criterion for the regularity of the extended Torelli map $\overline{M}_g \rightarrow \overline{A}_g^\tau$, and illustrate it in the case of the 2nd Voronoi compactification, as proved by Mumford and Namikawa. Then we reduce the cases of perfect cones, resp. central cones, to the existence of an \mathbb{R} -valued, resp. a \mathbb{Z} -valued emm for any graph of genus $\leq g$. We also prove that the existence of a strong \mathbb{R} -emm implies that $\overline{A}_g^{\text{perf}}$ and $\overline{A}_g^{\text{vor}}$ share a common open neighborhood of the image of \overline{M}_g .

In Section 4, we prove that a \mathbb{Z} -emm exists for any graph of genus $g \leq 6$, and does not exist for some explicit graphs of genus 9, thus settling negatively the extension question for the central cone compactification and $g \geq 9$.

In Section 5, we prove that a strong \mathbb{R} -emm exists for any graph, thus proving the regularity of $\overline{M}_g \rightarrow \overline{A}_g^{\text{perf}}$ and the statement about a common neighborhood.

In the concluding Section 6 we discuss some possible extensions of our results.

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1. Toroidal compactifications of A_g

Here, A_g stands for the moduli space of principally polarized abelian varieties. The theory of its toroidal compactifications over \mathbb{C} was developed by Mumford and his coworkers in [AMRT75]; [FC90] contains an extension to the arithmetic case, over \mathbb{Z} . It is parallel to the theory of ordinary toric varieties.

As in toric geometry, there are two dual lattices, M (for monomials) and N (for 1-parameter subgroups in the torus). The real vector space $N_{\mathbb{R}}$ is the ambient space for a fan τ , and $M_{\mathbb{R}}$ is the ambient space for polyhedra. For compactifications of A_g , one fixes a free abelian group $\Lambda \simeq \mathbb{Z}^g$. Then $M = \text{Sym}^2 \Lambda$, and $N = \Gamma^2 \Lambda^*$ is the dual abelian group, the second divided power of Λ^* .

Let us choose a basis f_1, \dots, f_g of Λ and a dual basis f_1^*, \dots, f_g^* of Λ^* , so that $(f_i^*, f_j) = \sigma_{ij}$. Then the elements of the lattice $M = \text{Sym}^2 \Lambda$ are integral homogeneous quadratic functions $q = \sum_{i \leq j} q_{ij} f_i f_j$, $q_{ij} \in \mathbb{Z}$, on Λ^* . These correspond to symmetric *half-integral* $g \times g$ matrices $A = (a_{ij})$, which means that $a_{ii} \in \mathbb{Z}$ and $a_{ij} \in \frac{1}{2}\mathbb{Z}$ for $i \neq j$. Equivalently, $2A$ is the matrix of an even integral bilinear form.

The elements of $N = \Gamma^2 \Lambda^*$ are integral tensors $\sum b_{ij} f_i^* \otimes f_j^*$ symmetric under the involution $f_i^* \otimes f_j^* \mapsto f_j^* \otimes f_i^*$. Thus, N can be identified with the space of symmetric *integral* matrices $B = (b_{ij})$, $b_{ij} \in \mathbb{Z}$.

Both M and N can be considered as the lattices in the space of real symmetric $g \times g$ -matrices. They are dual with respect to the inner product $(A, B) = \text{Tr } AB$.

Now let C be the open cone C in $N_{\mathbb{R}}$ consisting of positive-definite symmetric real matrices. This cone is self-dual with respect to the above inner product. One fixes its ‘‘closure’’ \overline{C} . To be precise, \overline{C} is the real cone spanned by semi definite positive symmetric matrices $B \geq 0$ with rational radical (i.e., the null space of B has to have a basis of vectors with rational coordinates).

Then a toroidal compactification \overline{A}_g^τ of A_g is defined by a fan τ (i.e. a collection of finitely generated rational cones, closed under taking faces) in $N_\mathbb{R}$ satisfying the following properties:

- (1) $\text{Supp } \tau = \overline{C}$.
- (2) The natural $\text{GL}(g, \mathbb{Z})$ -action on $N_\mathbb{R}$ sends cones of τ to cones of τ .
- (3) There are only finitely many orbits of cones under this action.

The following are three standard fans corresponding to three standard toroidal compactifications of A_g :

The perfect cones fan τ^{perf} , otherwise known as the *1st Voronoi fan*. The cones are defined to be the cones over the faces of the convex hull of $N \cap (\overline{C} \setminus 0)$. By a result of Barnes and Cohn [BC76], the vertices of $\text{Conv } N \cap (\overline{C} \setminus 0)$ (that is, the rays of τ^{perf}) are of the form a^{*2} , where $a^* = \sum a_i f_i^*$ is an integral primitive (i.e. indivisible) nonzero element of Λ^* . Thus, every perfect cone σ has the form $\sigma = \sum_s \mathbb{R}_{\geq 0} a_s^{*2}$ for some collection $\{a_s^*\} \subset \Lambda^* \setminus 0$.

For $q \in \text{Sym}^2 \Lambda_\mathbb{R}$, one has $(q, a^{*2}) = q(a^*)$, the value of the quadratic function q at the integral point $a^* \in \Lambda^*$. Thus, if σ^\vee is the dual cone in $M_\mathbb{R}$, then the elements of the interior $(\sigma^\vee)^0$ are the positive definite quadratic functions which attain the minimum on the same finite subset $\{a_s^*\}$.

In particular, for a maximal cone $\sigma \in \tau^{\text{perf}}$, the cone σ^\vee is generated by one quadratic function which is determined up to a multiple by the set of its minimal integral nonzero vectors. Such quadratic forms are called *perfect*, hence the name of this fan.

The second Voronoi fan τ^{vor} , sometimes referred to as Delaunay-Voronoi fan, or L -type decomposition. The locally closed cones τ^0 of this fan consist of quadratic forms which define the same Delaunay decomposition of $\Lambda_\mathbb{R}/\Lambda$.

The central cones fan τ^{cent} , corresponding to the normalization of the Igusa blowup. Let Q be the convex hull of $C \cap (M \setminus 0)$. This is an infinite polyhedron whose faces are (finite) polytopes. The fan τ^{cent} is the dual fan of Q . The vertices of $\text{Conv}(C \cap (M \setminus 0))$ are called *central quadratic forms*. Note that they are integral by definition. The corresponding cones of τ^{cent} are maximal-dimensional *central cones*.

Each of the fans τ^{perf} , τ^{vor} , τ^{cent} admits a strictly convex support function, (1) and (3) by definition and (2) by [Ale02]. Hence, the compactifications $\overline{A}_g^{\text{perf}}$, $\overline{A}_g^{\text{vor}}$, $\overline{A}_g^{\text{cent}}$ are projective by Tai's criterion [AMRT75, IV.2].

2. Graphs and quadratic forms

G will denote a graph with edges e_i , $i = 1, \dots, m$ and vertices v_j , $j = 1, \dots, n$. We allow multiple edges and loops. We fix an orientation of edges. Then we have the usual boundary homomorphism

$$\partial: C_1(G, \mathbb{Z}) = \oplus_i \mathbb{Z} e_i \rightarrow C_0(G, \mathbb{Z}) = \oplus_j \mathbb{Z} v_j, \quad \partial e_i = \text{end}(e_i) - \text{beg}(e_i)$$

The kernel of this map is the space of cycles $H_1(G, \mathbb{Z})$ and the cokernel is $H_0(G, \mathbb{Z})$. We will assume G to be connected, so that $H_0(G, \mathbb{Z}) = \mathbb{Z}$. Dually, we have the homomorphism

$$d: C^0(G, \mathbb{Z}) = \oplus_j \mathbb{Z} v_j^* \rightarrow C^1(G, \mathbb{Z}) = \oplus_i \mathbb{Z} e_i^*, \quad dv_j^* = \sum_{v_j = \text{end}(e_i)} e_i^* - \sum_{v_j = \text{beg}(e_i)} e_i^*$$

with kernel $H^0(G, \mathbb{Z}) = \mathbb{Z}$ and cokernel $H^1(G, \mathbb{Z})$.

Definition 2.1. We call the elements e_i^* in $H^1(G, \mathbb{Z})$ *coedges*, to distinguish them from the edges $e_i \in C_1(G, \mathbb{Z})$. Thus, coedges are cocycles and edges are chains.

Since any graph is homotopy equivalent to a graph with one vertex and g loops for some $g \geq 0$, called the *genus* of G , $H_1(G, \mathbb{Z})$ and $H^1(G, \mathbb{Z})$ are free abelian groups of rank g , dual to each other.

Lemma 2.2. *One has the following:*

- (1) *The elements e_i^* span $H^1(G, \mathbb{Z})$.*
- (2) *$e_i^* = 0$ iff the edge e_i is a bridge in G .*
- (3) *The graph G is a simple loop (a graph with one vertex and one edge) or is loopless and is 2-connected \iff the edges can not be divided into two disjoint groups $I_1 \sqcup I_2$ such that*

$$H^1(G, \mathbb{Z}) = \langle e_{i_1}^* \rangle \oplus \langle e_{i_2}^* \rangle, \quad i_s \in I_s.$$

Proof. (1) and (2) are obvious. For (3), consider a partition of edges $I_1 \sqcup I_2$, and denote by G_s , $s = 1, 2$, the graph formed by the edges of I_s . Note that the zero set of I_s in $H_1(G)$ is $H_1(G_{3-s})$. Since e_i^* span $H^1(G, \mathbb{Z})$, the condition of (3) is that the intersection is zero, equivalently that $H_1(G)$ is spanned by $H_1(G_1)$ and $H_1(G_2)$, i.e. every simple cycle in G lies entirely either in G_1 or in G_2 . If G has a loop (but G is not a loop itself) or G is not 2-connected, then obviously there is such a decomposition. Vice versa, given such a decomposition, every vertex in $G_1 \cap G_2$ is a cut of G or is a vertex of a loop, so G is not 2-connected or it has a loop. \square

The following lemma gives explicit \mathbb{Z} -bases for $H_1(G, \mathbb{Z})$ and $H^1(G, \mathbb{Z})$.

Lemma 2.3. *For a collection of edges e_i , $i \in I \subset \{1, \dots, m\}$, the following conditions are equivalent:*

- (1) *e_i^* form an \mathbb{R} -basis of $H^1(G, \mathbb{R})$.*
- (2) *e_i^* form a \mathbb{Z} -basis of $H^1(G, \mathbb{Z})$.*
- (3) *The complement of $\{e_i\}$ is a spanning tree T of G .*

If either of these conditions is satisfied then there exists a basis of $H_1(G, \mathbb{Z})$ of the form

$$f_i = e_i + \sum_{e_s \in T} b_{is} e_s, \quad b_{is} = 0, \pm 1, \quad i \in I.$$

Proof. Of course, (2) implies (1). Let us prove (1) \Rightarrow (3). Note that $|I| = g$.

By the Euler's formula, $g(G) = m + 1 - n$ and $\chi(G) = 1 - g$. Since the graph $G' = G \setminus \{e_i, i \in I\}$ has the same vertices and g fewer edges, we have $\chi(G') = 1$. Then either G' is connected and is a tree, or else G' is disconnected and has a nonzero loop, call it ℓ . Then for all $i \in I$ we have $e_i^*(\ell) = 0$, hence $\{e_i^*, i \in I\}$ is not a basis of $H^1(G, \mathbb{R})$. QED.

(3) \Rightarrow (2). We prove this by constructing a dual basis $\{f_i\}$ in $H_1(G, \mathbb{Z})$ to the set $\{e_i^*\}$. Since T is a tree, for each j there exists a unique path in T from the end to the beginning of e_i . In other words, there exists a unique $f_i \in H_1(G, \mathbb{Z})$ which can be written as

$$f_i = e_i + \sum_{e_s \in T} b_{is} e_s, \quad b_{is} = 0, \pm 1.$$

Then it is clear that $e_i^*(f_k) = 1$ if $j = k$ and 0 otherwise. Thus, $\{e_i^*\}$ and $\{f_i\}$ are dual bases in $H^1(G, \mathbb{Z})$ and $H_1(G, \mathbb{Z})$. \square

Definition 2.4. An *edge-minimizing metric*, abbreviated to *emm*, of a graph G is a quadratic form $q \in \text{Sym}^2 H_1(G)$ such that

- (1) $q > 0$, i.e. q is positive definite.
- (2) $q(e_i^*) = 1$ for each edge e_i which is not a bridge (i.e. for each $e_i^* \neq 0$).
- (3) $q(v^*) \geq 1$ for any $v^* \in H^1(G, \mathbb{Z}) \setminus 0$.

A *strong* edge-minimizing metric, in addition, satisfies the following: if $q(v^*) = 1$ for some $v^* \in H^1(G, \mathbb{Z})$ then $\pm v^*$ is a coedge.

In other words, q is a metric on the lattice $H^1(G, \mathbb{Z})$ and the nonzero $\pm e_i^*$ are among the shortest (resp. exactly the shortest) integral vectors in this metric.

We will distinguish between $q \in \text{Sym}^2 H_1(G, R)$, where R is \mathbb{Z} , \mathbb{Q} , or \mathbb{R} . We will call these \mathbb{Z} -emm, \mathbb{Q} -emm, \mathbb{R} -emm respectively. There will be no difference between \mathbb{Q} -emms and \mathbb{R} -emms for our purposes.

Definition 2.5. By Lemma 2.2(3), one has $H^1(G, \mathbb{Z}) = \oplus_k H^1(G_k, \mathbb{Z})$ for some graphs G_k so that each G_k is either a simple loop or loopless and 2-connected, and so that each nonzero e_i^* lies in one of the direct summands. We may call G_k *irreducible components* of G .

Lemma 2.6. *There exists a $(\mathbb{Z}, \mathbb{Q}, \text{ or } \mathbb{R})$ emm for a graph $G \iff$ there exist emms for each irreducible component G_k .*

Proof. The restriction q_k of an emm q to each $H^1(G_k, \mathbb{Z})$ is an emm. Vice versa, given emms q_k for graphs G_k , we can take $q = \sum q_k$ to be an emm for G . \square

Lemma 2.7. *To construct a $(\mathbb{Z}, \mathbb{Q}, \text{ or } \mathbb{R})$ emm for a graph G , it is sufficient to construct an emm for several related cubic bridgeless graphs.*

A remark concerning our terminology: *cubic* is the same as trivalent, and *bridgeless* is the same as 2-connected.

Proof. By the above Lemma, it is sufficient to construct an emm for each irreducible component G_k . If G_k is a loop then $q = x^2$ is a \mathbb{Z} -emm. So assume G_k is not a loop.

Removing vertices of degree 2 and replacing the adjacent two edges by a single edge results in reducing some duplication in the set $\{e_i^*\}$. Next, we inductively insert an edge into a vertex of degree ≥ 4 , until we get to a cubic graph G'_k . Doing so does not change H_1 but adds more vectors e_i^* , so the condition for G'_k is stronger than for G_k . \square

3. Criteria for the regularity of the extended Torelli map

As in Section 1, we fix a lattice $\Lambda \simeq \mathbb{Z}^g$, a fan τ , and a corresponding toroidal compactification \overline{A}_g^τ . We also consider a graph G of genus $a \leq g$ and write its homology as a quotient $\Lambda \twoheadrightarrow H_1(G, \mathbb{Z})$. This gives a cotorsion embedding of $N(G) := \Gamma^2 H^1(\Gamma, \mathbb{Z})$ into $N = \Gamma^2 \Lambda^*$. We work either over a field or over \mathbb{Z} .

Definition 3.1. We will denote by $S(G)$ the set of nonzero vectors e_i^{*2} in $N(G)$.

Theorem 3.2 (General criterion). *The Torelli map $\overline{M}_g \dashrightarrow \overline{A}_g^\tau$ is regular in a neighborhood of a stable curve $[C]$ iff for the dual graph $G(C)$ there exists a cone σ in the fan τ such that $S(G) \subset \sigma$.*

Proof. The stacks $(\overline{M}_g, \partial \overline{M}_g)$ and $(\overline{A}_g^\tau, \partial \overline{A}_g^\tau)$ are toroidal. The second one by definition, and the first one because it is a smooth stack of dimension $3g - 3$ and the boundary divisors have normal crossings.

Thus, in a neighborhood of a boundary point $[C]$ of \overline{M}_g corresponding to a stable curve, \overline{M}_g is a toroidal stack modeled on $(\mathbb{A}^1, 0)^m \times \mathbb{G}_m^{3g-3-m}$, where m is the number of edges of the dual graph Γ of C , and \mathbb{G}_m is the multiplicative group. This corresponds to a standard m -dimensional cone in \mathbb{R}^{3g-3} generated by the first m coordinate vectors which are in a bijection with the edges e_i of Γ .

By the Picard-Lefschetz monodromy formula, near the boundary the Torelli map is described by the linear map sending the vector e_i to $(e_i^*)^2 \in N$. We conclude the proof by applying a well-known criterion of regularity for toroidal varieties saying that the rational map is regular iff every cone of the first fan maps into a cone in the second fan.

The coarse moduli spaces are locally finite Galois quotients of appropriate toroidal neighborhoods for \overline{M}_g , $\overline{A}_g^{\text{vor}}$. The regularity of the rational map is unaffected by such Galois covers. Hence, the result for the coarse moduli spaces is the same as for the stacks. \square

Lemma 3.3. *The map $\overline{M}_g \dashrightarrow \overline{A}_g^\tau$ is regular on an open union of strata of \overline{M}_g .*

Proof. For two stable curves C, C' , the stratum of C is in the closure of the stratum of C' iff G is a contraction of G' . Then $H_1(G', \mathbb{Z}) \twoheadrightarrow H_1(G, \mathbb{Z})$, the lattice $N(G)$ is a cotorsion sublattice in $N(G')$, and $S(G) \subset S(G')$. Thus, if $S(G') \subset \sigma$ then $S(G) \subset \sigma$. $S(G)$ also lies in a cone of the induced fan on $N(G)_\mathbb{R}$. \square

As a first application, we reprove the following result of Mumford and Namikawa, cf. [Nam76, §18].

Theorem 3.4. *The map $\overline{M}_g \rightarrow \overline{A}_g^{\text{vor}}$ is regular.*

Proof. This immediately follows from Theorem 3.2, Lemma 2.3, and the following well known elementary fact about the “dicing” 2nd Voronoi cones, cf. [ER94]. \square

Lemma 3.5 (Dicings). *Let $v_i^* \in \Lambda^*$, $i \in I$, be finitely many nonzero primitive vectors. Then the following conditions are equivalent:*

- (1) $\{v_i^{*2}\}$ lie in the same 2nd Voronoi cone,
- (2) $\sum_{\mathbb{R}_{\geq 0}} v_i^{*2}$ is a 2nd Voronoi cone,
- (3) Any linearly independent subset $\{v_j^*\}$, $j \in J \subset I$, is a \mathbb{Z} -basis of $\Lambda^* \cap \sum \mathbb{R}v_j^*$.

Remark 3.6. The systems of vectors in the above lemma are known by various names: totally unimodular systems of vectors, dicings, regular matroids. The “dicing” refers to the corresponding Delaunay decomposition of $\Lambda_\mathbb{R}$ periodic w.r.t. Λ . It is given by “dicing” the vector space $\Lambda_\mathbb{R}$ by the parallel systems of hyperplanes $\{v_i^* = n_i \in \mathbb{Z}\}$.

Seymour’s classification theorem on regular matroids, which can be found in [Oxl92], says that all regular matroids are graphic, cographic, a special matroid R_{10} , or can be obtained from these by a sort of “tensor product”.

The regular matroids above, corresponding to $\{e_i^* \in H^1(G, \mathbb{Z})\}$, are the cographic matroids. This gives the combinatorial description of the toroidal Torelli map. By [ER94, 4.1] every dicing 2nd Voronoi cone is simplicial. Thus, the open neighborhood of $\text{im } \overline{M}_g$ in the stack $\overline{A}_g^{\text{vor}}$ corresponding to the cographic dicing cones has at worst abelian quotient singularities.

We now turn to the cases of perfect and central cones.

Theorem 3.7. (1) *The Torelli map $\overline{M}_g \dashrightarrow \overline{A}_g^{\text{perf}}$ is regular in a neighborhood of a stable curve $[C]$ iff the dual graph $G(C)$ has an \mathbb{R} -emm.*

- (2) Moreover, if every graph G of genus $\leq g$ has a strong \mathbb{R} -emm then $\overline{A}_g^{\text{perf}}$ and $\overline{A}_g^{\text{vor}}$ share a common open neighborhood of the image of \overline{M}_g .

Proof. (1) By the description of the perfect fan given in Section 1, e_i^{*2} lie in a perfect cone iff they are edges of some perfect cone σ . This means that there exists a positive definite quadratic form q such that the nonzero e_i^* are among the shortest integral vectors w.r.t. q . This is our definition of an \mathbb{R} -emm.

(2) By the above, the strata of $\overline{A}_g^{\text{vor}}$ corresponding to the cographic regular matroids gives a Zariski open neighborhood U of the image of \overline{M}_g . We want to show that each of these 2nd Voronoi cones is also a perfect cone. This means that there exists a $q > 0$ such that the nonzero $\pm e_i^*$ are exactly the shortest integral vectors w.r.t. q . This is our definition of a strong \mathbb{R} -emm. \square

Theorem 3.8. *The Torelli map $M_g \dashrightarrow \overline{A}_g^{\text{cent}}$ is regular in a neighborhood of a stable curve $[C]$ iff the dual graph $G(C)$ has a \mathbb{Z} -emm.*

Proof. Applying Theorem 3.2, if the map is regular then $\{e_i^{*2}\}$ lie in the same central cone σ , which we can pick to be maximal-dimensional. The corresponding dual cone σ^\vee is 1-dimensional and is spanned by a central form $q \in M$. This is an integral positive definite form characterized by the following property: for any $f \in \sigma$ and any other integral positive definite form $q' \in M$ one has $(q, f) \leq (q', f)$. Since every $e_i^* \neq 0$ is a primitive vector in Λ^* , there exist a q' with $(q', e_i^{*2}) = q'(e_i^*) = 1$. Therefore, $q(e_i^*) = 1$ for all nonbridge edges e_i , and so q is a \mathbb{Z} -emm.

Vice versa, if q is a \mathbb{Z} -emm of G then $1 = q(e_i^*) \leq q'(e_i^*)$ for any $q' \in M \cap C \setminus 0$, so $\{e_i^{*2}\} \subset \sigma$ for any cone σ^\vee containing q . \square

Lemma 3.9. *The subset of \overline{M}_g of curves admitting a \mathbb{Z} -emm, resp. \mathbb{R} -emm, is an open union of strata (cf. Lemma 2.7).*

Proof. Using the notations of the proof of Lemma 3.3, the restriction of an emm on $H^1(G', \mathbb{Z})$ to $H^1(G, \mathbb{Z})$ is an emm for G . \square

4. \mathbb{Z} -emms and positive cycle 2-covers of graphs

Lemma 4.1. *Let q be a \mathbb{Z} -emm of a graph G . Then the lattice $(H^1(G, \mathbb{Z}), 2q)$ is a direct sum of the standard root lattices A_n, D_n ($n \geq 4$), E_n ($n = 6, 7, 8$).*

If, in addition, G has no loops and is 2-connected, then the root lattice $(H^1(G, \mathbb{Z}), 2q)$ is irreducible, i.e. it is a single copy of A_n, D_n , or E_n .

Proof. The bilinear form $2q$ is integral, symmetric, positive definite, and even. The set R of vectors r with $r^2 = 2$ spans $H^1(G, \mathbb{Z})$ since it contains e_i^* . Thus, R is a simply laced root system, which must be a direct sum of A_n, D_n, E_n by a standard classification.

If $(H^1(G, \mathbb{Z}), 2q)$ is not irreducible then it splits as a direct sum $\langle e_{i_1}^* \rangle \oplus \langle e_{i_2}^* \rangle$ for some partition $I_1 \sqcup I_2$ of edges. Then G is not 2-connected loopless by Lemma 2.2. \square

Recall that a graph is called *projective planar* if it admits an embedding into \mathbb{RP}^2 . The main result of this section is the following.

Theorem 4.2. *Let G be a 2-connected loopless graph. Then*

- (1) G has a \mathbb{Z} -emm of type $A_g \iff G$ is planar.

(2) For $g \geq 4$, G has a \mathbb{Z} -emm of type $D_g \iff G$ is projective planar.
Moreover, the directions \Leftarrow hold for any graph.

Remark 4.3. The direction \Leftarrow of (1) is due to Namikawa [Nam73, Prop.5].

Corollary 4.4. Every graph of genus $g \leq 6$ admits a \mathbb{Z} -emm.

Proof. The well-known Kuratowski theorem says that a graph is not planar iff it contains a subgraph homeomorphic to K_5 or $K_{3,3}$. There is a similar theorem of Archdeacon [Arc80, Arc81] for projective planar graphs which has a much longer list of 103 minimal counterexamples. All of those graphs have genus ≥ 7 , except for a single graph G_1 . The existence of a \mathbb{Z} -emm for G_1 can be easily established by a direct, although quite lengthy, computation.

We note that [Arc80, Arc81] is concerned with graphs without loops and multiple edges. But a loop just adds a single \mathbb{Z} summand to $H_1(G, \mathbb{Z})$, and the multiple edges do not affect (projective) planarity. \square

Remark 4.5. By extending this method, [ALT+10] proves the existence of a \mathbb{Z} -emm for every graph of genus 7 and 8. This amounts to checking all the cubic genus 6 and 7 graphs from Archdeacon's list and the cubic graphs obtained from them by adding one or two edges.

Corollary 4.6. For any $g \geq 9$, there exists a graph G of genus g which does not admit a \mathbb{Z} -emm.

Proof. Since \mathbb{Z} -emms of type E_n only appear for $g = 6, 7, 8$, in genus $g \geq 9$ it is sufficient to take any 2-connected loopless graph containing a graph from Archdeacon's list [Arc80, Arc81] as a subgraph. For example, the graph of genus 9 in Figure 1 contains the minimal nonplanar graph of genus 6 from [Arc80, Arc81]: For $g > 9$, one can obtain G from it by

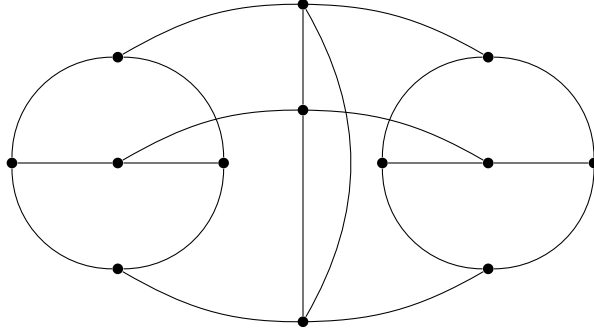


FIGURE 1. A graph of genus 9 without a \mathbb{Z} -emm

adding $g - 9$ edges. \square

Joe Tennini pointed out to us that the list in [Arc80, Arc81] contains a graph with 7 vertices. This implies that the complete graph K_n is not projective planar for $n \geq 7$ and does not admit a \mathbb{Z} -emm since $g(K_7) = 15$.

The basic idea of our proof of Theorem 4.2 is the following. We can assume that G is bridgeless, by contracting the bridges. A *cycle 2-cover* of G is a collection of cycles c_k

such that every edge appears in $\cup c_k$ exactly twice. (A long standing conjecture of Szekeres-Seymour says that every bridgeless graph has such a cycle 2-cover. We don't need the validity of this conjecture for our proof).

Now let us say $\{c_k\}$ is a cycle 2-cover, and consider the quadratic form $q = \frac{1}{2} \sum c_k^2$. Then q is integral and since every e_i^2 appears in q with coefficient 1, one has $q(e_i^*) = (q, e_i^{*2}) = 1$. However, in general q is only positive *semi* definite.

Definition 4.7. A cycle 2-cover $\{c_k, k = 1, \dots, N\}$ of a graph is called *positive* if the quadratic form $q = \frac{1}{2} \sum_{k=1}^N c_k^2$ is positive definite.

Cycle 2-covers are closely related to embeddings of graphs into closed topological surfaces. Given an embedding $G \hookrightarrow S^2$, resp. $G \hookrightarrow \mathbb{RP}^2$, we will construct a \mathbb{Z} -emm of type A_g , resp. D_g , on $H^1(G, \mathbb{Z})$. Then we will prove the converse by using the fact that the quadratic forms A_n and D_n can be written as sums of $\geq n$ squares of integral linear forms.

So let $\{c_k\}$ be a cycle 2-cover. Divide each c_k into a sum of simple (not repeating vertices) cycles d_ℓ . For each d_ℓ , take a copy D_ℓ of a 2-disk and identify its boundary with d_ℓ . Glue these disks along the edges e_i . The result is a closed surface X which may have isolated singular points at some vertices, as follows.

For a vertex v , consider a simple cycle d_ℓ in the given 2-cover that goes through v . We constructed X by gluing the boundary of a disk D_ℓ to d_ℓ . In X there is a neighboring disk that also contains v ; continue from disk to disk until you have made a full circle around v . If these are *all* the d_ℓ containing v then X is smooth at v . In general, there will be several such full circles, and so X is obtained from a smooth closed surface \tilde{X} by gluing together several points to such bad vertices v .

Theorem 4.8. Let G be a loopless 2-connected graph admitting a cycle double cover $\{c_k\}$. Then $\{c_k\}$ is positive $\iff X = S^2$ or \mathbb{RP}^2 .

Proof. Assume that $q > 0$. First, we claim that X has no singular points, i.e. that $X = \tilde{X}$. Assuming the opposite, let X' be the surface obtained from X by normalizing at a single singular point v , and let G' be the preimage of G in X' . Since G is 2-connected, G' is a connected graph. Since G' has the same edges as G but more vertices, we have $g(G') < g(G)$, and $H_1(G') \subset H_1(G)$ is a proper subspace. But q is the same sum of squares of elements of $H_1(G')$, so it can not be positive definite.

Next, the Euler characteristic of the smooth surface X is

$$\chi(X) = N - E + V = N - (E - V + 1) + 1 = N - g + 1$$

Since q is positive definite, $N \geq g$. Hence, $\chi(X) \geq 1$. There are only two smooth closed surfaces with $\chi \geq 1$: S^2 ($\chi = 2$, $N = g + 1$) and \mathbb{RP}^2 ($\chi = 1$, $N = g$).

Now, for the opposite direction. If $X = S^2$ then G divides the sphere into $g + 1$ regions with boundaries c_k . These obviously generate $H_1(G, \mathbb{Z})$, so $q > 0$.

In the case $X = \mathbb{RP}^2$, let $\pi : S^2 \rightarrow \mathbb{RP}^2$ be the 2:1 cover, and let $G' = \pi^{-1}(G)$. Because $\pi_* H_1(G', \mathbb{R}) = H_1(G, \mathbb{R})$ and the cycles on S^2 generate $H_1(G', \mathbb{R})$, the cycles c_k generate $H_1(G, \mathbb{R})$. Hence, $\sum c_k^2$ is positive definite. \square

Proof of Theorem 4.2. (1) Let G be an arbitrary graph with an embedding $G \subset S^2$. If G is not connected, we add bridges to make it connected. Then the set $S^2 \setminus G$ is a union of $g + 1$ regions bounded by the cycles c_k which can be given compatible orientations. Then $H_1(G, \mathbb{Z}) = \mathbb{Z}^{g+1} / \mathbb{Z} \sum c_k$, and the dual lattice $H^1(G, \mathbb{Z})$ is the hyperplane $\{(n_k) \in \mathbb{Z}^{g+1} \mid \sum n_k = 0\}$. The quadratic form $2q = \sum c_k^2$ is the restriction of the standard Euclidean form on \mathbb{Z}^{g+1} to $H^1(G, \mathbb{Z})$. This is the standard definition of the A_g lattice. Each nonbridge edge

e_i belongs to precisely two 2-cells k_1, k_2 . Then $\pm e_i^* = c_{k_1}^* - c_{k_2}^*$, where $\{c_k^*\}$ is the Euclidean basis of \mathbb{Z}^{g+1} .

Vice versa, suppose that a 2-connected loopless graph G has a \mathbb{Z} -emm of type A_g . Note that the quadratic form of A_g is a sum of $g+1$ squares of integral linear functions:

$$2q = 2 \sum_{i=1}^g x_i^2 - 2 \sum_{i=1}^{g-1} x_i x_{i+1} = x_1^2 + (x_1 - x_2)^2 + \dots + (x_{g-1} - x_g)^2 + x_g^2$$

Consider the $g+1$ cycles c_k in $H_1(G, \mathbb{Z})$ corresponding to these linear terms. We claim that for each edge e_i of G we have $c_k(e_i^*) = 0$ or ± 1 . Indeed, if $|c_k(e_i^*)| \geq 2$ then for $q = \frac{1}{2} \sum_k c_k^2$ we have $q(e_i^*) \geq 2$, which contradicts our assumption $q(e_i^*) = 1$. Thus, for each edge there exist exactly two cycles with $c_k^2(e_i^*) = 1$, the collection of c_k is a cycle 2-cover. By the proof of the previous Theorem 4.8 the corresponding ambient surface is S^2 .

(2) Let $G \subset \mathbb{RP}^2$ be an arbitrary graph. Again, we make it connected, if necessary, by adding bridges. Let $G' \subset S^2$ be the preimage under the 2:1 cover $S^2 \rightarrow \mathbb{RP}^2$.

If G' is disconnected then it has two components both isomorphic to G . Then G is planar, (1) applies, and an embedding $G \hookrightarrow S^2$ defines a \mathbb{Z} -emm of type A_g . Consider the $g+1$ regions $S^2 \setminus G$ and the corresponding cycles c_k . Since $g+1 \geq 5$ and the complete graph K_5 is non-planar, there exist two cycles c_{k_1}, c_{k_2} which do not share an edge. Then the quadratic form $\frac{1}{2} \sum_{k \neq k_1, k_2} c_k^2 + \frac{1}{2} (c_{k_1} - c_{k_2})^2$ is a \mathbb{Z} -emm of G , and it is easy to check that it has type D_g .

Otherwise, G' is a connected graph of genus $2g-1$. As above, $H^1(G', \mathbb{Z})$ together with the quadratic form $\sum_{k=1}^{2g} c_k^2$ is a root system of type A_{2g-1} in its realization as the hyperplane $\{\sum n_k = 0\}$ in \mathbb{Z}^{2g} .

On the homology, the antipodal involution ι can be written as $c_k \mapsto -c_{k+g}$, where we set $c_m := c_{m-2g}$ if $m > 2g$. The image of $H_1(G', \mathbb{Z})$ in $H_1(G, \mathbb{Z})$ is the projection of $H_1(G', \mathbb{Z})$ onto the $(+1)$ -eigenspace $H_1^+(G', \mathbb{R})$, and can be identified with the standard Euclidean \mathbb{Z}^g with the basis c_1, \dots, c_g . The homology group $H_1(G, \mathbb{Z})$ is the \mathbb{Z}_2 -extension of it obtained by adding vector $\frac{1}{2}(1, \dots, 1)$.

Thus, the dual lattice $(H^1(G, \mathbb{Z}), 2q)$ can be identified with the sublattice of \mathbb{Z}^g of integral vectors with even sum of coordinates. This is the standard definition of the D_g lattice.

For the opposite direction, note that the quadratic form for D_g is a sum of g squares:

$$\begin{aligned} 2q &= 2 \sum_{i=1}^g x_i^2 - 2x_1x_3 - 2x_2x_3 - 2 \sum_{i=3}^{g-1} x_i x_{i+1} = \\ &= (x_1 + x_2 - x_3)^2 + (x_1 - x_2)^2 + (x_3 - x_4)^2 + \dots + (x_{g-1} - x_g)^2 + x_g^2 \end{aligned}$$

This gives g cycles c_k and thus a positive double cover by g or $g+1$ simple cycles d_ℓ , which defines an embedding of G into S^2 or \mathbb{RP}^2 , if G is 2-connected and loopless. Finally, if G is planar then it is moreover projective planar. \square

5. Existence of \mathbb{R} -emms

Recall that by Lemma 2.7 it is sufficient to construct \mathbb{R} -emms for cubic graphs. We begin by characterizing coedges. The following simple lemma will be useful:

Lemma 5.1. *If G is a connected bridgeless graph with the property that every edge is contained in a two element cutset then G is cyclic.*

Proof. Suppose, as the induction hypothesis, G is a minimal counterexample. Take any edge e and a cutset $\{e, f\}$ containing it. This exhibits G as a "cycle" $[e, G_a, f, G_b]$ with G_a and G_b disjoint and joined only by e and f . Enlarge this cycle to exhibit G as a larger "cycle", or "necklace" consisting of a chain of "gems" G_1, G_2, \dots, G_r connected cyclically by single edges. Further assume that this necklace is maximal, so that no G_i contains a G_i -bridge. Then by the induction hypothesis, each G_i must be cyclic. Since we assumed G was not cyclic, one of them contains an edge. That edge is not in any 2-element G -cutset, contradicting the choice of G . \square

We can now give the promised characterization of coedges:

Definition 5.2. A cycle in a graph Γ will be called a $(0, 1)$ -cycle if all directed edges appear in it with coefficients in $\{+1, 0, -1\}$; that is, the cycle is a sum of simple cycles with disjoint edge supports.

Lemma 5.3. A nonzero cocycle $z \in H^1(\Gamma, \mathbb{Z})$ is a coedge $\iff z(c) \in \{+1, 0, -1\}$ for all $(0, 1)$ -cycles c .

Proof. Clearly a coedge satisfies this condition, so we need to prove the converse.

We can assume that Γ is 2-edge connected, i.e. connected and bridgeless. In a bridgeless graph, all edges are divided into equivalence classes by $e \sim e'$ iff $e^* = \pm e'^*$. By contracting all but one edge in each equivalence class, we can assume that Γ is 3-edge connected.

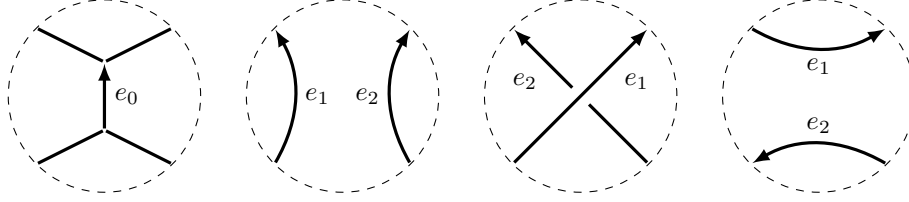
We will proceed by induction on the number of edges in Γ . Choose some edge e_1 of Γ . We form a new graph $\Gamma \setminus e_1$ by deleting e_1 . We note that $(0, 1)$ -cycles in $\Gamma \setminus e_1$ are $(0, 1)$ -cycles in Γ , and we have the natural pullback map $f : H^1(\Gamma, \mathbb{R}) \rightarrow H^1(\Gamma \setminus e_1, \mathbb{R})$, so $f(z)$ satisfies the conditions of the lemma and by induction is a coedge if it is nonzero. But $\ker(f) = e_1^*$, so either $z = ne_1^*$ (in which case we immediately have $z = \pm e_1^*$, a coedge) or $z = ne_1^* + e_2^*$. In this case we claim that $n \in \{-1, 0, 1\}$. To show this we exhibit a simple cycle c in $\Gamma \setminus e_2$ containing e_1 . This is certainly possible as long as e_1 is not a bridge in $\Gamma \setminus e_2$. On the other hand, if e_1 is a bridge in $\Gamma \setminus e_2$ then $e_1^* = \pm e_2^*$ in $H^1(\Gamma, \mathbb{Z})$, so $f(z) = 0$, and $z = \pm e_1^*$ as above. If $n = 0$ we're done, so can assume that $z = e_1^* + e_2^*$ by changing the orientation of e_1 if necessary.

If $\Gamma \setminus \{e_1, e_2\}$ has a bridge e_3 then we get a 3-term relation on coedges $e_3^* = e_1^* \pm e_2^*$ in Γ , implying either $z = e_3^*$, in which case we're done, or $z = 2e_2^* - e_3^*$, which would contradict $z(c) \in \{+1, 0, -1\}$ for simple cycles c , by an argument similar to that given above. The alternative is that $\Gamma \setminus \{e_1, e_2\}$ is bridgeless, which we show is impossible.

Assume $\Gamma \setminus \{e_1, e_2\}$ is bridgeless. Delete any edge $e \neq e_1, e_2$. Induction tells us that z becomes a coedge e_3^* in $\Gamma \setminus e$, so we have a four term relation $e_1^* + e_2^* = e_3^* + ke$ in Γ . Thus we have a four element cutset $\{e_1, e_2, e, e_3\}$ on Γ , and so a two element cutset $\{e, e_3\}$ on $\Gamma \setminus \{e_1, e_2\}$ for any e . But by the previous lemma 5.1, the only graphs where every edge is contained in a two element cutset are cyclic graphs, and if $\Gamma \setminus \{e_1, e_2\}$ was a cyclic graph we could easily find a $(0, 1)$ -cycle c in Γ such that $(e_1^* + e_2^*)(c) = 2$, a contradiction. \square

Theorem 5.4. Any bridgeless cubic graph G admits a strong \mathbb{Q} -emm.

Proof. We will reduce the problem to the existence of strong \mathbb{Q} -emms on certain strictly smaller graphs. In this way, we get an inductive construction of such forms. Let e_0 be an edge in G . We can produce 3 graphs on fewer vertices by modifying the region of G containing e_0 as shown on Figure 2. Note that G_1 and G_2 are analogous to each other. We call the two edges formed by this process e_1 and e_2 in each of G_1, G_2, G_3 , and we orient them as shown.

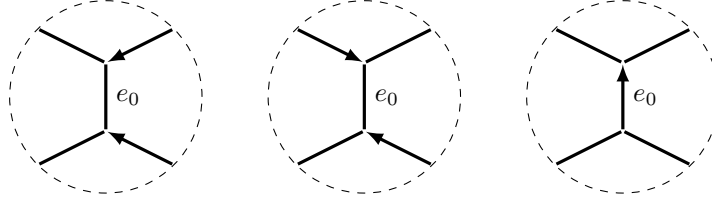
FIGURE 2. The graphs G, G_1, G_2, G_3

Since integral cycles in G_i lift to integral cycles in G , we have maps $H_1(G_i, \mathbb{Z}) \rightarrow H_1(G, \mathbb{Z})$, $i \in \{1, 2, 3\}$. Hence we get the opposite maps $\phi_i : H^1(G, \mathbb{Z}) \rightarrow H^1(G_i, \mathbb{Z})$. We note the following:

Claim 5.5. For a cocycle $z \in H^1(G, \mathbb{Z})$, if $\phi_i(z)$ is a coedge for each $i \in \{1, 2, 3\}$ then z is itself a coedge.

Proof. This follows immediately from the Lemma 5.3 and the observation that every $(0, 1)$ -cycle in G corresponds to a $(0, 1)$ -cycle in at least one of the G_i \square

Examining these maps more closely, we see that the kernels of ϕ_i are generated by the cocycles shown in Figure 3. We also have maps on quadratic forms $\psi_i : \text{Sym}^2 H_1(G_i, \mathbb{Z}) \rightarrow \text{Sym}^2 H_1(G, \mathbb{Z})$. By the induction hypothesis we have strong \mathbb{Q} -emms q_i on G_i . These lift to forms $\psi_i(q_i)$ on G , positive semidefinite and zero only on $\ker(\phi_i)$. We wish to build a strong emm as a convex combination $x_1\psi_1(q_1) + x_2\psi_2(q_2) + x_3\psi_3(q_3)$ where $x_1 + x_2 + x_3 = 1, x_i \geq 0$.

FIGURE 3. Generators of $\ker \phi_1, \ker \phi_2, \ker \phi_3$

Assume first that G_i are bridgeless (the special cases where some of the G_i are not bridgeless will be dealt with later). Note that for every edge $e \neq e_0$ we have $\psi_i(q_i)(e^*) = 1$ because ϕ_i maps coedges to coedges. For every other (that is noncoedge) integral cocycle z not in $\ker(\phi_i)$ we have $\psi_i(q_i)(z) \geq 1$, with at least one of the $\psi_i(q_i)(z) > 1$ by Claim 5.5.

Hence if $x_i \neq 0$ for all i , we need only to verify the emm conditions on the three cocycles generating $\ker(\phi_i)$. For brevity we write $c_i = q_i(e_1^* + e_2^*)$ and note that since q_i is a quadratic form with $q_i(e_1^*) = q_i(e_2^*) = 1$ we must have $q_i(e_1^* - e_2^*) = 4 - c_i$. Hence, if $e_1^* \pm e_2^* \neq 0$ in G_i then $c_i \in [1, 3]$, with $c_i \in (1, 3)$ if $e_1^* \pm e_2^*$ are not coedges. We can now write the (strong) emm conditions as follows:

- (1) $x_2(4 - c_2) + x_3(4 - c_3) \geq 1$ when $\ker(\phi_1) \neq \{0\}$ and with equality only when the generator of $\ker(\phi_1)$ is a coedge.
- (2) $x_1(4 - c_1) + x_3c_3 \geq 1$ when $\ker(\phi_2) \neq \{0\}$ and with equality only when the generator of $\ker(\phi_2)$ is a coedge.
- (3) $x_1c_1 + x_2c_2 = 1$ ($\ker(\phi_3)$ is generated by a coedge, namely e_0^* .)

Note that these inequalities are symmetric except for the symbols $\geq, =$. Consider the generic case where $c_1, c_2, c_3 \in (1, 3)$ (we'll deal with the nongeneric cases later). We use three solution types, depending on the values of c_1, c_2, c_3 :

Case 1. When $(c_2 \leq 2 \text{ or } c_3 \leq 2)$ and $(c_1 \leq 2 \text{ or } c_3 \geq 2)$ set $x_1 = x_2 = \frac{1}{c_1 + c_2}$. Consider the left hand side of the first inequality, call it k . We have $k = \frac{4 - c_2}{c_1 + c_2} + \frac{(c_1 + c_2 - 2)(4 - c_3)}{c_1 + c_2}$. Since $c_3 < 3$ we have $k > \frac{2 + c_1}{c_1 + c_2} \geq 1$ when $c_2 \leq 2$. If $c_3 \leq 2$ then $k \geq \frac{2c_1 + c_2}{c_1 + c_2} > 1$. The other half of the conjunction follows by symmetry.

Case 2. When $c_1 \geq 2$ and $c_3 < 2$, set $x_2 = \epsilon c_1, x_1 = 1/c_1 - \epsilon c_2$ for small $\epsilon > 0$. Indeed, if we take $\epsilon = 0$ the left hand side of the first inequality simply becomes $\frac{(c_1 - 1)(4 - c_3)}{c_1}$, which is greater than 1. The second inequality is also satisfied (see case 1). Of course this doesn't quite work, since $x_2 = 0$, but by continuity the conditions still hold for suitably small ϵ .

Case 3. When $c_2 \geq 2$ and $c_3 > 2$, set $x_1 = \epsilon c_2, x_2 = 1/c_2 - \epsilon c_1$. This works by symmetry with case 2.

Hence all the generic cases are solved. Consider now the exceptional cases. That is, assume that either at least one of the G_i has a bridge, or that one of the c_i is 0, 1, 3, or 4. By listing all possible such cases, we will show that they are of only three types (up to symmetry), shown in Figure 4.

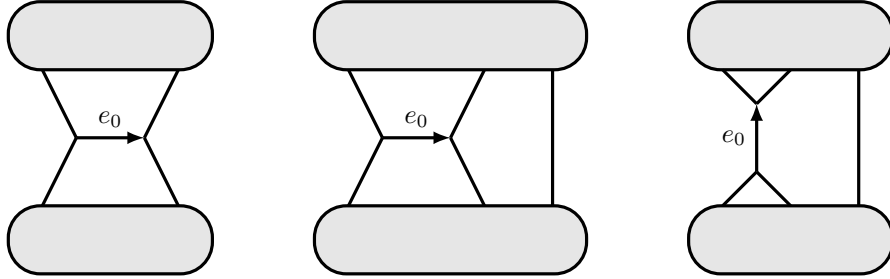


FIGURE 4. Exceptional cases A, B, C

- (1) Some $c_i \in \{0, 4\}$
 - (a) $c_1 = 0$ or $c_2 = 0$. This cannot happen in a bridgeless graph.
 - (b) $c_3 = 4$. This occurs only in graphs of type A.
 - (c) $c_3 = 0$. This is symmetric to $c_3 = 4$
 - (d) $c_1 = 4$. This happens exactly when $c_3 = 0$.
 - (e) $c_2 = 4$. This happens exactly when $c_3 = 4$.
- (2) Some G_i contains a bridge.
 - (a) G_1 contains a bridge. This is the situation of type B.
 - (b) G_2 contains a bridge. Symmetric to G_1 containing a bridge.
 - (c) G_3 contains a bridge. This is type C.
- (3) Some $c_i \in \{1, 3\}$. That is, one of $e_1^* \pm e_2^*$ is a coedge.
 - (a) $c_2 = 1$. This is type C.

- (b) $c_2 = 3$. This is type B.
- (c) $c_1 \in \{1, 3\}$. This is symmetric to $c_2 \in \{1, 3\}$.
- (d) $c_3 = 3$. This is type B.
- (e) $c_3 = 1$. This is symmetric to $c_3 = 3$.

If we choose e_0 to not lie in any 2-element cutset, then it becomes unnecessary to deal with case C. This is always possible, since by Lemma 5.1 every edge is part of a two element cutset only if G is cyclic. We proceed to construct strong emms for the remaining two cases:

(A) We observe that $\ker(\phi_1) = \{0\}$. Moreover, since every $(0,1)$ cycle in G is of the form $\phi_1(c)$ for some $(0,1)$ cycle c in G_1 , by the Lemma 5.3 every noncoedge in $H^1(G, \mathbb{Z})$ maps through ϕ_1 to a noncoedge in $H^1(G_1, \mathbb{Z})$. Hence $\psi_1(q_1)$ satisfies the conditions for a strong emm.

(B) We have $c_2 = c_3 = 3$; we construct the form $\frac{1}{3}\psi_2(q_2) + \frac{2}{3}\psi_3(q_3)$. We need to show that this is greater than 1 on all integral noncoedges. Equivalently (by Lemma 5.3), we need to show that for any $z \in H^1(G, \mathbb{Z})$ with $z(c) > 1$ for some $(0,1)$ -cycle c there is an $i \in \{2, 3\}$ and $(0,1)$ -cycle c' in G_i such that $\phi_i(z)(c') > 1$.

We may assume that c is the sum of at most two simple cycles (if $z(\sum c_i) > 1$ then $z(c_1) > 1$ or $z(c_1 + c_2) > 1$ for some c_1, c_2). Since c is a $(0,1)$ cycle it must be the image of a cycle in at least one of G_1, G_2, G_3 . If it is the image of a cycle in G_2 or G_3 there is nothing to show, so we can assume that it is the image of some cycle k in G_1 , but not of any cycle in G_2 or G_3 . In order for this to be true c must contain the edge e_0 and not the edge that becomes a bridge in G_1 .

By symmetry assume that k contains the edge e_1 and not e_2 . Note that since c was a sum of 2 (possibly trivial) simple cycles, so is k , say $k = k_1 + k_2$ where k_1 the simple summand containing e_1 . Note that, since k_2 contains neither e_1 nor e_2 , the image of k_2 in G is also the image of simple cycles in G_2 and G_3 , so if $\phi(z)(k_2) > 1$ the problem would be solved immediately. Hence we will further assume $\phi(z)(k_1) \neq 0$.

We will proceed to use k to construct a $(0,1)$ cycle k' in G_1 that contains both e_1 and e_2 and has $\phi_1(z)(k') > 1$. If we can succeed in doing this the result will follow, because depending on the relative orientations of e_1 and e_2 in k' there must be either a $(0,1)$ cycle c' in G_2 that maps to the same cycle in G as k' does (so $\phi_2(z)(c') = \phi_1(z)(k') > 1$) or one in G_3 that maps to the same cycle as k' (so $\phi_3(z)(c') > 1$).

To build k' , find a simple cycle l passing through e_2 that intersects k_2 in a (possibly empty) arc (continuous path of edges). Such a cycle always exists, since given any cycle containing e_2 (these exist by connectedness) there is a maximal arc (possibly the whole cycle) disjoint from k_2 , and a (possibly empty) arc in k_2 joining its endpoints. We can choose the arc in k_2 such that $l + k_2$ is also a $(0,1)$ cycle. But now since $\phi_1(z)(k) > 1$, out of the four $(0,1)$ cycles $k_1 \pm l$, $k_1 \pm (l + k_2)$ one of $\phi_1(z)(k_1 \pm l)$, $\phi_1(z)(k_1 \pm (l + k_2))$ must be greater than 1. (Indeed, recalling our assumption that $\phi_1(z)(k_1) \neq 0$, if $\phi_1(z)(l) \neq 0$ one of $k_1 \pm l$ will work, if $\phi_1(z)(k_2) \neq 0$ one of $k_1 \pm (l + k_2)$ will work, otherwise all four work.) Calling this cycle k' the result follows.

Hence the nongeneric cases are resolved. This concludes the proof of Theorem 5.4. \square

Remark 5.6. In the induction argument of Theorem 5.4, it may happen that some of the incoming and outgoing edges are in fact the same. Then one of the graphs G_i may have

a component which is a “loop with one edge e and zero vertices”. We deal with this case formally, by taking e^2 to be the corresponding quadratic form.

Theorem 5.7. *Any graph G admits a strong \mathbb{Q} -emm.*

Proof. Existence of an \mathbb{Q} -emm follows at once from Theorem 5.4 and Lemma 2.7 which reduces graph G to a disjoint collection $G' = \sqcup G'_k$ of cubic graphs. However, the inclusion $S(G) \subset S(G')$ (see Definition 3.1) may be strict, so a strong \mathbb{Q} -emm q of G' may not be a strong \mathbb{Q} -emm of G .

By the result [ER94, 4.1] which we mentioned in Remark 3.6, the distinct vectors e_i^{*2} for the graph G' are linearly independent. Thus, there exists a $q_0 \in M_{\mathbb{Q}}$ such that q_0 is zero on $S(G)$ and positive on $S(G') \setminus S(G)$. Then $q + \epsilon q_0$ for $0 < \epsilon \ll 1$ is a required strong \mathbb{Q} -emm for G . \square

6. Concluding remarks and generalizations

6.1. Characterization of \mathbb{Z} -emms of type E_n . It would be interesting to find a geometric characterization of graphs admitting \mathbb{Z} -emms of types E_6 , E_7 , E_8 , similar to the characterization for A_n and D_n given in Theorem 4.2.

6.2. Special quadratic forms, and physical interpretation. For any collection of positive real numbers $(\lambda_1, \dots, \lambda_m)$ there is a natural positive definite quadratic form $Q = \sum \lambda_i e_i^{*2}$ on the homology group $H_1(G, \mathbb{R})$. Since Q is nondegenerate, we can use it to identify H_1 and H^1 , thus producing a positive definite quadratic form q on H^1 . In coordinates, the matrix of q is the inverse of the matrix of Q . Searching for an \mathbb{R} -emm of this form leads to a system of m nonlinear equations in m variables which seems to be hard to solve. We note that our solution for an \mathbb{R} -emm is not of this special form.

One can make a graph into an electric network by putting resistors λ_i along the edges e_i . The *total energy dissipation* of this electric system is Q . The condition $q(e_i^*) = 1$ can be reformulated in these terms as follows. For any edge e_i , let G_i be the graph obtained by cutting the edge e_i in the middle, thus producing two end points p_i, q_i . Then the condition is that the resistance of G_i between the points p_i and q_i is 1, for each $i = 1, \dots, m$.

6.3. All dicing 2nd Voronoi cones. The method of the proof of Theorem 5.4 may also apply to arbitrary, not necessarily cographic, regular matroids. This would give a bigger open set in which $\overline{A}_g^{\text{perf}}$ and $\overline{A}_g^{\text{vor}}$ coincide.

6.4. Other Torelli maps. The proofs of Theorems 3.4, 3.7, 3.8 work in a more general situation, if we replace $S(G)$ by any finite set $\{v^{*2}\}$ of symmetric rank 1 tensors. Thus, they give regularity criteria for any rational map $\overline{M} \dashrightarrow \overline{A}_g^\tau$, $\tau = \tau^{\text{vor}}, \tau^{\text{perf}}, \tau^{\text{cent}}$, for as long as \overline{M} is toroidal and the monodromy map has a specific form $r_i \mapsto v_i^{*2}$. For example, once properly set up, this may apply to intermediate jacobians of cubic 3-folds.

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